

Free coherent states and p -adic numbers

S.V.Kozyrev

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Abstract

Free coherent states for a system with two degrees of freedom is defined. Existence of the homeomorphism of the ring of integer 2-adic numbers to the set of coherent states corresponding to an eigenvalue of the operator of annihilation is proved. It is shown that the metric of free Fock space induces the 2-adic topology on the set of coherent states.

Free (or Boltzmannian) Fock space has been considered in some recent works on quantum chromodynamics [1], [2], [3] and noncommutative probability [4], [5].

The subject of this work is free coherent states. We will introduce free coherent states and investigate the metric space of coherent states corresponding to a fixed eigenvalue of the operator of annihilation. It will be shown that archimedean metric of Hilbert space induce non-archimedean metric on the set of free coherent states. The main result of present paper is the construction of the homeomorphism from the ring of integer 2-adic numbers to the space of free coherent states with topology defined by the Hilbert metric. We will consider the system with two degrees of freedom. The system with one degree of freedom was investigated in [6].

The free commutation relations are particular case of q -deformed relations

$$A_i A_j^\dagger - q A_j^\dagger A_i = \delta_{ij}$$

with $q = 0$. A correspondence of q -deformed commutation relations and non-archimedean (ultrametric) geometry was discussed in [7]. Non-archimedean mathematical physics was studied in [8].

Free coherent states lies in the free Fock space. Free (or Boltzmannian) Fock space F over a Hilbert space H is the completion of the tensor algebra

$$F = \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$

Creation and annihilation operators are defined in the following way:

$$A^\dagger(f) f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n$$

$$A(f) f_1 \otimes \dots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \dots \otimes f_n$$

⁰ e-mail: kozyrev@class.mian.su, Address: OSV, Inst. Chem. Phys., 117334, Kosygina 4, Moscow

where $\langle f, g \rangle$ is the scalar product in the Hilbert space H . Scalar product τ in the free Fock space is defined by the standard construction of the direct sum of tensor products of Euclidean spaces.

We consider the case $H = C \oplus C$, where C is the field of complex numbers. In this case we have two creation operators A_0^\dagger, A_1^\dagger and two annihilation operators A_0, A_1 with commutation relations

$$A_i A_j^\dagger = \delta_{ij}. \quad (1)$$

The vacuum vector Ω in the free Fock space satisfies

$$A_i \Omega = 0. \quad (2)$$

We define free coherent states in the following way.

Let us fix two positive constants $\gamma_0, \gamma_1, 0 < \gamma_i < 1$. Let us consider an infinite sequence of indices $U = u_1 u_2 u_3 \dots, u_i = 0, 1$.

The free coherent state X_U is introduced as the series

$$X_U = \sum_{k=0}^{\infty} X_k.$$

Here $X_0 = \Omega$ is vacuum and $X_k = \gamma_{u_k} A_{u_k}^\dagger X_{k-1}$. The norm $\|X_k\|$ satisfies the condition

$$\|X_k\| = \prod_{i=1}^k \gamma_{u_i}$$

with $0 < \gamma_i < 1$. Therefore the series X_U converges.

Free coherent states are eigenvectors of the annihilation operator

$$\gamma_0^{-1} A_0 + \gamma_1^{-1} A_1;$$

for eigenvalue 1, i.e.

$$(\gamma_0^{-1} A_0 + \gamma_1^{-1} A_1) X_U = X_U \quad \forall U.$$

Degeneration of this eigenspace is parametrized by the set $\{U\}$ of infinite sequences of 0 and 1.

These sequences have a natural interpretation as 2-adic numbers. Every sequence U is in one-to-one correspondence with a 2-adic number $\sum_{i=1}^{\infty} u_i 2^{i-1}$.

Let us consider the following metric on the set of free coherent states. Let U, V be arbitrary sequences of 0 and 1. These sequences coincide up to an element with number k . Let X_U, X_V be corresponding free coherent states. We introduce the metric ρ :

$$\rho(X_U, X_V) = \|X_k\| = \prod_{i=1}^k \gamma_{u_i}.$$

The metric ρ is an ultrametric. This means that the metric ρ obeys the strong triangle inequality

$$\rho(X_U, X_V) \leq \max(\rho(X_U, X_W), \rho(X_V, X_W))$$

for arbitrary U, V, W . Let us investigate the topology on the set of free coherent states generated by the metric ρ . Topology in a metric space is defined by the set of all closed balls in this space. The set of closed balls $\{B_{U,k}\}$ in the metric space of free coherent states with metric ρ is parametrized by pairs (U, k) . Here U is a sequence of indices, k is non-negative integer and the ball $B_{U,k}$ contains all coherent states X_V where U and V coincide up to the index with number k .

The map $X_U \mapsto \sum_{i=1}^{\infty} u_i 2^{i-1}$ sends the ball $B_{U,k}$ to a closed ball of radius 2^{-k} with a center in $\sum_{i=1}^{\infty} u_i 2^{i-1}$ in the ring of integer 2-adic numbers.

Therefore the metric $\rho(X_U, X_V)$ induces the topology of the ring of integer 2-adic numbers on the set of free coherent states. The map $X_U \mapsto \sum_{i=1}^{\infty} u_i 2^{i-1}$ is homeomorphism.

If $\gamma_0 = \gamma_1 = \gamma$, then $\rho(X_U, X_V) = \gamma^k$ is the metric on 2-adic disc.

Let us compare the metric ρ with the usual metric τ of the scalar product in the free Fock space. It is easy to see that

$$\tau(X_U, X_V) = \|X_k\| \left(\sum_{i=k+1}^{\infty} \left(\prod_{j=k+1}^i \gamma_{u_j}^2 + \prod_{j=k+1}^i \gamma_{v_j}^2 \right) \right)^{\frac{1}{2}}.$$

Let $\gamma_0 < \gamma_1$. Then

$$\left(\frac{2\gamma_0^2}{1-\gamma_0^2} \right)^{\frac{1}{2}} \rho \leq \tau \leq \left(\frac{2\gamma_1^2}{1-\gamma_1^2} \right)^{\frac{1}{2}} \rho;$$

and metrics τ and ρ generate equivalent topologies on the space of free coherent states. We have proved the following theorem.

Theorem. *The map $X_U \mapsto \sum_{i=1}^{\infty} u_i 2^{i-1}$ is homeomorphism of the set of free coherent states with topology generated by the metric of scalar product $\tau(X_U, X_V)$ to the ring of integer 2-adic numbers.*

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